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# MULTI-VELOCITY SERBER-WILSON NEUTRON DIFFUSION CALCULATIONS



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# MULTI-VELOCITY SERBER-WILSON NEUTRON DIFFUSION CALCULATIONS

Certain types of neutron diffusion calculations were considerably simplified when the Serber-Wilson Method was introduced about eight years ago. This method, semi-empirical in nature and named after its co-discoverers,[1] was first formulated for the one-velocity isotropic theory and applied to spherical geometries. Within these limits it has in general proved to be a fairly accurate method. If restricted to the source-free case it has, in addition, turned out to be quite manageable both analytically and numerically.

The Serber-Wilson Method was, however, not extensively used here until about three years ago. At that time the computation techniques involved were systematized and somewhat improved.[2] A year later a set of special function tables were completed resulting in a considerable saving of computing time.[3] The work involved was further shortened when the CPC calculator was brought into the picture about a year ago.

Let us consider neutron diffusion problems under the above restrictions for the moment. The corresponding mathematical description is then furnished by the integro-differential equation below:

$$(1)\quad \left[\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} + \sigma\right]\mathcal{N}(r,\mu) = \frac{1}{2}\sigma c\,\mathcal{N}(r),$$

1. LA-234 by R. Serber, BM-441 by A. H. Wilson.
2. LA-756 by B. Carlson.
3. LA-1364, 1365, 1366 by B. Carlson, M. Goldstein, and D. Sweeney.

where $\mathcal{N}(r, \mu)$ denotes the neutron flux as a function of radius and direction cosine and $\mathcal{N}(r)$ the integral of $\mathcal{N}(r, \mu)$ over $\mu$ from -1 to +1. The quantities $\sigma$ and c in (1) represent known step-functions of r characterizing the assembly of media under consideration, the general medium being, in this case, a concentric spherical shell. Specifically, $\sigma$ is the inverse mean free path for neutrons, and c the number of neutrons emerging per collision.

The following steps are involved in the Serber-Wilson Method and may, in fact, be regarded as a definition of the method:

(A) Prescribing $\mathcal{N}(r)$ for the general medium with an analytical expression involving two arbitrary constants.

(B) Defining and deriving two functionals of $\mathcal{N}(r)$, having the dimension of $\mathcal{N}(r)$, one depending perhaps on the geometry and the other being the net neutron flux.

(C) Applying a sufficient number of physical conditions, primarily continuity conditions, on the two functionals to determine the arbitrary constants.

An approximate or asymptotic expression for $\mathcal{N}(r)$ may be obtained either by applying the Spherical Harmonic transformation to (1) or by studying the integral equation equivalent to (1).[1] In either case we obtain:

$$(2) \quad \mathcal{N}(r) \sim A\left[\frac{\sin kr}{kr} + \bar{A}\,\frac{\cos kr}{|k|r}\right] = A^{*}\,\frac{\sin k(r+r_0)}{kr} ,$$

1. LA-247 by K. M. Case, LA-571 by B. Carlson. See also Appendix, p. 27.

[redacted]

with $A = A^* \cos kr_o$, $\bar{A} = (|k|/k) \tan kr_o$, and k from the transcendental equation $k/\sigma = c \text{ art } (k/\sigma)$. The quantity k, taken to be positive, may be either real ($c \geq 1$) or pure imaginary ($1 > c \geq 0$).

For the spherical geometry which is being considered the inward radial flux $\overleftarrow{\mathcal{N}}(r) = 2\mathcal{N}(r,-1)$ was chosen as one of the functionals, the net flux $\bar{\mathcal{N}}(r) = \int_{-1}^{1} \mu \mathcal{N}(r,\mu) d\mu$ being the other. Differential equations for $\overleftarrow{\mathcal{N}}(r)$ and $\bar{\mathcal{N}}(r)$ are readily obtained from (1) and the solutions are immediate. For if we let $\mu = -1$ in (1) we have on the one hand:

$$(3) \quad \left[ -\frac{d}{dr} + \sigma \right] \overleftarrow{\mathcal{N}}(r) = \sigma c \; \mathcal{N}(r),$$

and hence:

$$(4) \quad \overleftarrow{\mathcal{N}}(r) = - e^{\sigma r} \int^{r} \sigma c \; \mathcal{N}(r') e^{-\sigma r'} dr' .$$

On the other hand (1) may be written in the form:

$$(5) \quad \left[ \mu \frac{\partial}{\partial r} + \left(\sigma + \frac{2\mu}{r}\right) + \frac{1}{r} \frac{\partial}{\partial \mu} (1-\mu^2) \right] \mathcal{N}(r,\mu) = \frac{1}{2} \sigma c \; \mathcal{N}(r),$$

which, integrated over $\mu$ from -1 to +1, gives

$$\left[ \frac{d}{dr} + \frac{2}{r} \right] \bar{\mathcal{N}}(r) = \sigma (c-1) \mathcal{N}(r),$$



[redacted]

[redacted]

and hence:

$$(6) \qquad \bar{\mathcal{N}}(r) = \frac{1}{r^2}\int^r \sigma\,(c-1)r'^2\,\mathcal{N}(r')dr' \quad .$$

Substituting (2) in (4) we have:

$$(7) \quad \overleftarrow{\mathcal{N}}(r) = A\left[\,Q(|k|r,\phi) + \bar{A}\,R(|k|\,r,\phi)\right] =$$

$$= \frac{A\sigma c}{2k}\left\{i\left[E_1((\sigma+ik)r) - E_1((\sigma-ik)r)\right] + \frac{k\bar{A}}{|k|}\left[E_1((\sigma+ik)r) + E_1((\sigma-ik)r)\right]\right\}e^{\sigma r},$$

where $\phi = \mathrm{art}(k/\sigma)$, $c \geq 1$, and $\phi = \mathrm{arth}\,|k|/\sigma$, $1 > c \geq 0$.

The functions Q and R are tabulated in LA-1364, LA-1365, and LA-1366, as are the functions S and T in the formula for $\bar{\mathcal{N}}(r)$. The latter is obtained by substituting (2) in (6):

$$(8) \quad \bar{\mathcal{N}}(r) = A\left[\,S(|k|\,r) + \bar{A}\,T(|k|\,r)\right] =$$

$$= \frac{A\sigma(c-1)}{k}\left[\frac{\sin kr - kr\cos kr}{(kr)^2} + \frac{k}{|k|}\,\bar{A}\,\frac{\cos kr + kr\sin kr}{(kr)^2}\right]$$

The Serber-Wilson Method may be extended to the anisotropic case and to geometries such as plane and cylindrical.[1] For the anisotropic case the transcendental equation for $k/\sigma$ will be different. For other geometries a substitute for the functional $\overleftarrow{\mathcal{N}}(r)$ may have to be found. And again, if a source function is present on the right-hand side of (1) it may be difficult to find an asymptotic expression for $\mathcal{N}(r)$. Generalizations in the above directions have on the whole

1. Transport Theory of Neutrons (LT-18) by B. Davison



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proved feasible, whereas, in the direction of more velocity groups serious difficulties have been encountered.

Let us then turn to the multi-velocity isotropic theory with G velocity groups. Instead of (1) we have the following equations where g is the group index, g = 1, 2, ... G:

$$(9)\quad \left[\mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial\mu} + \sigma_g\right]\mathcal{N}_g(r,\mu) = \frac{1}{2}\sum_{h=1}^{G}\sigma_h c_{gh}\,\mathcal{N}_h(r),$$

In the above expression $\sigma_g$ are the separate inverse mean free paths and $c_{gh}$ the transfer coefficients. Denoting the group velocities by $v_g$, $c_{gh}$ represents the number of neutrons of velocity $v_g$ emerging per collision of neutron of velocity $v_h$. $\sigma_g$ as well as $c_{gh}$ are calculated from measured cross-sections.

Applying the same principals to (9) as to the one-velocity case we find the following asymptotic form for the flux distributions:

$$(10)\quad \mathcal{N}_g(r) \sim \sum_{i=1}^{G}\alpha_g^i\,\mathcal{N}_i(r) = \sum_{i=1}^{G}\alpha_g^i A_i\left[\frac{\sin k_i r}{k_i r} + \bar{A}_i\frac{\cos k_i r}{|k_i| r}\right],$$

where $k_i$ are the eigenvalues and $\{\alpha_g^i\}$ the eigenvectors of the matrix equation:

$$(11)\quad \begin{pmatrix} c_{11} - \dfrac{k_i/\sigma_1}{\text{art}(k_i/\sigma_1)} & c_{12} \cdots\cdots\cdots & c_{1G} \\ c_{21} & c_{22} - \dfrac{k_i/\sigma_2}{\text{art}(k_i/\sigma_2)} \cdots\cdots & c_{2G} \\ \cdot & \cdot & \\ c_{G1} & c_{G2} \cdots \quad c_{GG} - \dfrac{k_i/\sigma_G}{\text{art}(k_i/\sigma_G)} & \end{pmatrix} \begin{pmatrix} \alpha_1^i \\ \dfrac{\sigma_2}{\sigma_1}\alpha_2^i \\ \cdot \\ \dfrac{\sigma_G}{\sigma_1}\alpha_G^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix}$$



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After solving (11) for $k_i$ and $\left\{\alpha_g^i\right\}$ taking $\alpha_i^i$ equal to unity, (10) is determined except for the arbitrary constants $\overleftarrow{A}_i$ and $\bar{A}_i$. This is remedied by introducing 2G functionals of (10), $\overleftarrow{\mathcal{N}}_g(r)$ and $\bar{\mathcal{N}}_g(r)$, and requiring these to be continuous at the boundaries. Applying the methods of pp 5-6 to equation (9), $\overleftarrow{\mathcal{N}}_g(r)$ and $\bar{\mathcal{N}}_g(r)$ are readily obtained. We have:

$$(12)\begin{cases} \overleftarrow{\mathcal{N}}_g(r) = 2\,\mathcal{N}_g(r,-1) = -e^{\sigma_g r}\int^r \sum_{h=1}^{G} \sigma_h c_{gh}\,\mathcal{N}_g(r')e^{-\sigma_g r'}\,dr', \\ \bar{\mathcal{N}}_g(r) = \int_{-1}^{1}\mu\,\mathcal{N}_g(r,\mu)\,d\mu = \frac{1}{r^2}\int^r r'^2 \sum_{h=1}^{G}\sigma_h(c_{gh}-\delta_{gh})\,\mathcal{N}_g(r')\,dr', \end{cases}$$

where $\delta_{gh}$ equal to unity if $h=g$ and zero if $h\neq g$.

The functionals (12) can be considerably simplified if we substitute (10) on the right-hand side and then make use of the following consequence of (11):

$$(13)\quad \sum_{h=1}^{G}\sigma_h c_{gh}\,\alpha_h^i = \sigma_g\,\alpha_g^i\,\frac{k_i/\sigma_g}{\operatorname{art}(k_i/\sigma_g)} = \sigma_g\,\alpha_g^i c_g^i \;,$$

where the last equality serves to define $c_g^i$. Performing these substitutions we obtain:

$$(14)\begin{cases} \overleftarrow{\mathcal{N}}_g(r) = -e^{\sigma_g r}\int^r \sigma_g \sum_{i=1}^{G} c_g^i\alpha_g^i\,\mathcal{N}_i(r')e^{-\sigma_g r'}\,dr' \;, \\ \bar{\mathcal{N}}_g(r) = \frac{1}{r^2}\int^r r'^2\sigma_g \sum_{i=1}^{G}(c_g^i-1)\,\alpha_g^i\,\mathcal{N}_i(r')\,dr' \;, \end{cases}$$

where the functions $\mathcal{N}_i(r)$ are those defined by (10).



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If $c_{gh} = 0$ for $g > h$ we have as a rule G distinct eigenvalues and consequently as many arbitrary constants as boundary conditions. We may, therefore, conclude that in this case the above generalization of the Serber-Wilson Method is valid. If, on the other hand, we are dealing with fissionable materials, for which $c_{gh} \neq 0$ for all $g > h$, we can, as a rule, not count on (11) to give as many as G eigenvalues. The result is that we are left with more boundary conditions than arbitrary constants.[1] A number of schemes have been proposed, none of them entirely satisfactory, which in one way or another circumvent the above difficulty. A new method which may in the end prove satisfactory will be introduced below. In the very few applications made to date it has turned out to be both accurate and practical.

In this new method we replace the quantities $c_{gh}$ in (11) by $\overline{c_{gh}}$ defined below, thus transforming $(c_{gh})$ into a right-triangular matrix:

$$(15) \qquad \begin{cases} \overline{c_{gh}} = 0, & g > h \\ \overline{c_{gh}} = c_{gh} - c_{hg}\dfrac{N_g}{N_h}, & g < h, \\ \overline{c_{gg}} = \displaystyle\sum_{h=g}^{G} c_{hg} + \sum_{h=1}^{g-1} c_{gh}\frac{N_h}{N_g} & . \end{cases}$$

This obviously eliminates the difficulties referred to but requires some explanations. Before turning to these, however, and defining $N_g$,

1. See Transport Theory of Neutrons (LT-18) by B. Davison, pp 180-185.



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the following consequences of (5) may be noted. If $c_{gh} = 0$ for $g > h$ then $\overline{c_{gh}} = c_{gh}$. Equation (11) as modified by (15) will in general have G eigenvalues $k_i$, $i = 1, 2, \ldots G$, $(k_i/\sigma_i) = \overline{c_{ii}} \cdot \mathrm{art}(k_i/\sigma_i)$. The elements of the eigenvectors $\{\alpha_g^i\}$ can be obtained successively (starting with $\alpha_i^i = 1$) since the determinant is right-triangular, i.e., has zeros below the diagonal, and $\alpha_g^i = 0$, $g > i$.

For the purpose of illustrating the above formulae and notation we consider for the moment the three-velocity case: Consequently, $k_1/\sigma_1 = (c_{11}+c_{21}+c_{31})$ art $k_1/\sigma_1$, $k_2/\sigma_2 = (c_{22}+c_{32}+c_{21}\frac{N_1}{N_2})$art $k_2/\sigma_2$, and $k_3/\sigma_3 = (c_{33}+c_{31}\frac{N_1}{N_3} + c_{32}\frac{N_2}{N_3})$ art $k_3/\sigma_3$. Also $(\alpha_1^1, \alpha_2^1, \alpha_3^1) = (1,0,0)$, $(\alpha_1^2, \alpha_2^2, \alpha_3^2) = \left(\frac{\sigma_2\bar{c}_{12}}{\sigma_1(c_1^2-\bar{c}_{11})}, 1, 0\right)$, and $(\alpha_1^3, \alpha_2^3, \alpha_3^3) = \left(\frac{\sigma_3\left[\bar{c}_{12}\bar{c}_{23}+\bar{c}_{13}(c_2^3-\bar{c}_{22})\right]}{\sigma_1(c_1^3-\bar{c}_{11})(c_2^3-\bar{c}_{22})}, \frac{\sigma_3\bar{c}_{23}}{\sigma_2(c_2^3-\bar{c}_{22})}, 1\right)$. The flux distributions for a central core (for which $\bar{A}_i = 0$) are then given by:

$$
(16)\quad \begin{cases}
\mathcal{N}_1(r) \sim A_1 \dfrac{\sin k_1 r}{k_1 r} + \alpha_1^2 A_2 \dfrac{\sin k_2 r}{k_2 r} + \alpha_1^3 A_3 \dfrac{\sin k_3 r}{k_3 r}, \\[2ex]
\mathcal{N}_2(r) \sim A_2 \dfrac{\sin k_2 r}{k_2 r} + \alpha_2^3 A_3 \dfrac{\sin k_3 r}{k_3 r}, \\[2ex]
\mathcal{N}_3(r) \sim A_3 \dfrac{\sin k_3 r}{k_3 r}.
\end{cases}
$$



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Due to the triangular character of the determinants and expressions above, the labor involved in finding eigenvalues and applying boundary conditions is considerably reduced. As an example, connect the six functionals of (16) with the corresponding expressions for an infinite shell. We find then that the resulting simultaneous equations can be grouped, in this case into three sets of two each, thus reducing the computational work.

Going back to (11), i.e., to the definition of $\overline{c_{gh}}$, it is evident that we are tampering with the interchange of neutrons. Studying groups #1 and #2, for instance, we find that for each collision in #1, $c_{21}$ neutrons are given to #2. Hence, if these are given to #1 rather than #2, as is done in (15), then #2 should receive some compensation. Letting $N_2/N_1$ denote the number of collisions in groups #2 per collision in group #1, we should clearly reduce $c_{12}$ (what #2 gives to #1) by $c_{21} \cdot N_1/N_2$. This ritual is performed for each pair of velocity groups and for each medium. However, since $N_g$ is obtained as an integral over $\mathcal{N}_g(r)$ and $\mathcal{N}_g(r)$ is not available until the boundary conditions have been applied, we are faced with 2G simultaneous equations, transcendental in half of the unknowns involved. The method requires, therefore, a rather elaborate trial and error procedure. It is an exact method only if the ratios $N_g/N_h$ are independent of r within each medium.

Using (14) we have, corresponding to (16), the following expressions for $\overleftarrow{\mathcal{N}}_g(r)$ and $\overrightarrow{\mathcal{N}}_g(r)$:



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$$
(17)\quad
\begin{cases}
\overleftarrow{\mathcal{N}}_1(r) = A_1 Q\left(k_1 r, \dfrac{k_1}{\sigma_1 c_1^1}\right) + \alpha_1^2 A_2 Q\left(k_2 r, \dfrac{k_2}{\sigma_1 c_1^2}\right) + \alpha_1^3 A_3 Q\left(k_3 r, \dfrac{k_3}{\sigma_1 c_1^3}\right) \\[2ex]
\overleftarrow{\mathcal{N}}_2(r) = A_2 Q\left(k_2 r, \dfrac{k_2}{\sigma_2 c_2^2}\right) + \alpha_2^3 A_3 Q\left(k_3 r, \dfrac{k_3}{\sigma_2 c_2^3}\right) \\[2ex]
\overleftarrow{\mathcal{N}}_3(r) = A_3 Q\left(k_3 r, \dfrac{k_3}{\sigma_3 c_3^3}\right)
\end{cases}
$$

and

$$
(18)\quad
\begin{cases}
\overrightarrow{\mathcal{N}}_1(r) = A_1 \dfrac{\sigma_1}{k_1}(c_1^1 - 1) S(k_1 r) + \alpha_1^2 A_2 \dfrac{\sigma_1}{k_2}(c_1^2 - 1) S(k_2 r) + \alpha_1^3 A_3 \dfrac{\sigma_1}{k_3}(c_1^3 - 1) S(k_3 r) \\[2ex]
\overrightarrow{\mathcal{N}}_2(r) = A_2 \dfrac{\sigma_2}{k_2}(c_2^2 - 1) S(k_2 r) + \alpha_2^3 A_3 \dfrac{\sigma_2}{k_3}(c_2^3 - 1) S(k_3 r) \\[2ex]
\overrightarrow{\mathcal{N}}_3(r) = A_3 \dfrac{\sigma_3}{k_3}(c_3^3 - 1) S(k_3 r)
\end{cases}
$$

The above formulae can easily be extended to G groups and to the general spherical shell. In solving systems involving expressions like (14) and (15) and M separate spherical media, we start with the 2(M-1) equations involving $A_G$, solve for these unknowns and proceed to the 2(M-1) equations involving $A_G$ and $A_{G-1}$, etc.

With the above method G-velocity problems are essentially reduced to G one-velocity problems each of which (for the proper values of $N_g$) must give the same result for the required critical parameter. The



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sine part of the formulae for $N_g$ for three velocity groups and a spherical shell of inner and outer radii $a_1$ and $a_2$ are given by:

$$
(19)\quad \left\{
\begin{aligned}
N_1(r) &= A_1 \frac{\sigma_1}{k_1} S(k_1 r) + \alpha_1^2 A_2 \frac{\sigma_1}{k_2} S(k_2 r) + \alpha_1^3 A_3 \frac{\sigma_1}{k_3} S(k_3 r) \Bigg]_{a_1}^{a_2} \\
N_2(r) &= A_2 \frac{\sigma_2}{k_2} S(k_2 r) + \alpha_2^3 A_3 \frac{\sigma_2}{k_3} S(k_3 r) \Bigg]_{a_1}^{a_2} \\
N_3(r) &= A_3 \frac{\sigma_3}{k_3} S(k_3 r) \Bigg]_{a_1}^{a_2}
\end{aligned}
\right.
$$



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## EXAMPLE I

Consider an untamped Oralloy sphere of density 18.8 gr/cm$^3$ described by the 3-velocity parameters given in LA-1276:

$$(20)\quad v_g = \begin{Bmatrix} 6 \\ 12 \\ 24 \end{Bmatrix}, \quad \sigma_g = \begin{Bmatrix} .3853 \\ .2408 \\ .1879 \end{Bmatrix}, \quad c_{gh} = \begin{Bmatrix} .8412 & .3682 & .3902 \\ .1862 & .6925 & .4725 \\ .2081 & .2783 & .5804 \end{Bmatrix}.$$

We propose to calculate the critical radius of the sphere and the three flux distributions using the method described above. These are then to be compared with the results in LA-1272, obtained by the Integral Theory Method.

We take as a first trial, $N_1/N_3 = 2.6$ and $N_2/N_3 = 1.4$; as a second trial 2.6 and 1.45; and as a final trial 2.7 and 1.4. By calculation the following table is obtained:

| CASE | I | II | III |
|---|---|---|---|
| Calculated Quantities | $N_1/N_3$=2.6<br>$N_2/N_3$=1.4 | $N_1/N_3$=2.6<br>$N_2/N_3$=1.45 | $N_1/N_3$=2.7<br>$N_2/N_3$=1.4 |
| $\bar{c}_{11}, \bar{c}_{12}, \bar{c}_{13}$ | 1.2355, .0224, -.1509 | 1.2355, .0343, -.1509 | 1.2355, .0091, -.1717 |
| $\bar{c}_{22}, \bar{c}_{23}$ | 1.3166, .0829 | 1.3047, .0690 | 1.3299, .0829 |
| $\bar{c}_{33}$ | 1.5111 | 1.5250 | 1.5319 |
| $k_1, k_2, k_3$ | .35319, .26290, .27656 | .35319, .25692, .28142 | .35319, .26952, .28381 |
| $c_1^1, c_1^2, c_1^3$ | 1.2355, 1.1396, 1.1530 | 1.2355, 1.1339, 1.1578 | 1.2355, 1.1460, 1.1602 |
| $c_2^2 \quad c_2^3$ | 0 1.3166, 1.3442 | 0 1.3047, 1.3542 | 0 1.3299, 1.3591 |
| $c_3^3$ | 0 0 1.5111 | 0 0 1.5250 | 0 0 1.5319 |



[redacted]

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Table (continued)

| CASE | I | | | II | | | III | | |
|---|---|---|---|---|---|---|---|---|---|
| $\alpha_1^1, \alpha_1^2, \alpha_1^3$ | 1 | -.1460 | .4943 | 1 | -.2110 | .6470 | 1 | -.0635 | .9447 |
| $\alpha_2^2, \alpha_2^3$ | 0 | 1 | 2.3438 | 0 | 1 | 1.0877 | 0 | 1 | 2.2153 |
| $\alpha_3^3$ | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 |

Taking $A_3 = 1$ and solving $\overleftarrow{\mathcal{N}}_3(r) = Q(k_3 a_1, k_3/\sigma_3 c_3^3) = 0$ for $a_1$ we find: Case I: $a_1 = 8.489$, Case II: $a_1 = 8.323$, and Case III: $a_1 = 8.243$. In solving $\overleftarrow{\mathcal{N}}_3(r) = 0$ and calculating the quantities below we make use of one of the Serber-Wilson Tables, in this case LA-1364.

Continuing the work, denoting $k_i/\sigma_g c_g^i$ by $\phi_g^i$, likewise $Q(k_i a_1, \phi_g^i)$ by $Q_{ig}$, and $S(k_i a_1)$ by $S_i$, we have:

| CASE | I | II | III |
|---|---|---|---|
| $\phi_1^1, \phi_1^2, \phi_1^3$ | .7419, .5987, .6225 | .7419, .5881, .6308 | .7419, .6104, .6349 |
| $\phi_2^2, \phi_2^3$ | - .8292, .8544 | - .8178, .8630 | - .8416, .8672 |
| $\phi_3^3$ | - - .9740 | - - .9821 | - - .9860 |
| $k_1 a_1, k_2 a_1, k_3 a_1$ | 2.9982, 2.2318, 2.3477 | 2.9396, 2.1383, 2.3422 | 2.9113, 2.2217, 2.3394 |
| $S_1 \quad S_2 \quad S_3$ | .34600, .43355, .42800 | .35648, .43580, .42833 | .36134, .43389, .42849 |
| $Q_{11} \quad Q_{21} \quad Q_{31}$ | -.12417, .13861, .08812 | -.11204, .17882, .08781 | -.10588, .13907, .08768 |
| $Q_{22} \quad Q_{32}$ | .07418, .02780 | .11167, .02757 | .07454, .02749 |
| $Q_{33}$ | .00011 | .00002 | .00002 |



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Table (continued)

| CASE | I | II | III |
|---|---|---|---|
| $\frac{\sigma_3}{\sigma_2}\frac{N_2}{N_3} - \alpha_2^3$ | -1.25136 | .04376 | -1.12286 |
| $\frac{\sigma_3}{\sigma_1}\frac{N_1}{N_3} - \alpha_1^3$ | .77364 | .62094 | .37201 |
| $A_2$ | -1.1743 | .0393 | -1.0531 |
| $A_1$ | .9335 | .9503 | .4438 |
| $\overleftarrow{\mathcal{N}}_1(a_1)$ | -.04860 | -.05115 | .04514 |
| $\overleftarrow{\mathcal{N}}_2(a_1)$ | -.02196 | .03438 | -.01760 |

We use the above results for $\overleftarrow{\mathcal{N}}_1$ and $\overleftarrow{\mathcal{N}}_2$ (which should be equal to zero for the correct trial combination) to interpolate for $N_1/N_3$ and $N_2/N_3$. Linear interpolation is in this case equivalent to solving the equations:

$$(21)\quad \begin{cases} \mathcal{N}_1 \approx -.04860 + \frac{.09374}{.10}\left(\frac{N_1}{N_3} - 2.6\right) - \frac{.00255}{.05}\left(\frac{N_2}{N_3} - 1.4\right) = 0, \\ \mathcal{N}_2 \approx -.02196 + \frac{.00436}{.10}\left(\frac{N_1}{N_3} - 2.6\right) + \frac{.05634}{.05}\left(\frac{N_2}{N_3} - 1.4\right) = 0, \end{cases}$$

simultaneously. The solution of (21) gives $N_1/N_3 = 2.653$ and $N_2/N_3 = 1.417$ from which, by calculation, we find $k_2 = .26430$, $k_3 = .28207$, $a_1 = 8.301$,



[redacted]

[redacted]

$\alpha_1^2 = -.1295$, $\alpha_1^3 = .7568$, $\alpha_2^3 = 1.6882$, $A_1 = .6918$, and $A_2 = -.5378$.

This problem was also solved using the two-velocity parameters of Example II. Result: $a_1 = 8.315$. Furthermore, a variation of the method was tried, making $\bar{c}_{12}$ rather than $\bar{c}_{21}$ equal to zero. The result in this case: $a_1 = 8.320$.

The following table gives a comparison of the Serber-Wilson Method and the Integral Theory Method:

| Theory | $a_1$ | | | $N_1/N_3$ | $N_2/N_3$ |
|---|---|---|---|---|---|
| | 1-vel.* | 2-vel. | 3-vel. | 3-vel. | 3-vel. |
| S.W. | 8.39 | 8.32 | 8.30 | 2.653 | 1.417 |
| I.T. | 8.72 | 8.70 | 8.70 | 2.630 | 1.416 |

*Parameters (LA-1276): $\sigma = .2821$, $c = 1.2936$.

The flux densities (as functions of r) do not agree nearly as well. Cf. graph on page 24 and Table VII (Second Set) in LA-1272.



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EXAMPLE II

We consider next an Oralloy (Oy) sphere of density 18.8 gr/cm$^3$ tamped by an infinite Tuballoy (Tu) shell of density 19.0 and look for the critical radius and the flux distributions. To simplify the work here we content ourselves with a two-velocity calculation. In LA-1276 we find the following parameters for Oy and Tu:

$$(22)\quad \begin{cases} \text{Oy:}\quad v_g = \begin{Bmatrix} 6.43 \\ 19.6 \end{Bmatrix}, \quad \sigma_g = \begin{Bmatrix} .365 \\ .197 \end{Bmatrix}, \quad c_{gh} = \begin{Bmatrix} .897 & .553 \\ .347 & .863 \end{Bmatrix}, \\ \text{Tu:}\quad v_g = \begin{Bmatrix} 6.43 \\ 19.6 \end{Bmatrix}, \quad \sigma_g = \begin{Bmatrix} .365 \\ .197 \end{Bmatrix}, \quad c_{gh} = \begin{Bmatrix} .98 & .62 \\ .00 & .50 \end{Bmatrix}. \end{cases}$$

For the flux distributions we write according to (10):

$$(23)\quad \text{Oy:} \begin{cases} \mathcal{N}_1(r) = A_1 \dfrac{\sin k_1 r}{k_1 r} + \alpha_1^2 A_2 \dfrac{\sin k_2 r}{k_2 r}, \\ \mathcal{N}_2(r) = A_2 \dfrac{\sin k_2 r}{k_2 r}, \end{cases}$$

$$(24)\quad \text{Tu:} \begin{cases} \mathcal{N}_1(r) = B_1 \dfrac{e^{-k_1 r}}{k_1 r} + \alpha_1^2 B_2 \dfrac{e^{-k_2 r}}{k_2 r}, \\ \mathcal{N}_2(r) = B_2 \dfrac{e^{-k_2 r}}{k_2 r}. \end{cases}$$

Note that $v_g$, $\sigma_g$, $c_{gh}$, $k_i$, $c_g^i$, and $\alpha_g^i$ in general are functions of the medium, although for the sake of simplicity, we omit notations to this effect.



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The boundary conditions are described by the following equations, where the left-hand side refers to the Oy and the right-hand side to the Tu:

$$
(25)\quad \left\{
\begin{array}{lrcl}
 & \underline{\text{Core}} & & \underline{\text{Tamper}} \\
\text{(a)} & A_1 Q(k_1 a_1, k_1/\sigma_1 c_1^1) + \alpha_1^2 A_2 Q(k_2 a_1, k_2/\sigma_1 c_1^2) & = & B_1 R_t(\sigma_1 a_1, k_1/\sigma_1) + \alpha_1^2 B_2 R_t(\sigma_1 a_1, k_2/\sigma_1), \\
\text{(b)} & A_1 \frac{\sigma_1}{k_1}(c_1^1-1)S(k_1 a_1) + \alpha_1^2 A_2 \frac{\sigma_1}{k_2}(c_1^2-1)S(k_2 a_1) & = & B_1 \frac{\sigma_1}{k_1}(c_1^1-1)T_t(k_1 a_1) + \alpha_1^2 B_2 \frac{\sigma_1}{k_2}(c_1^2-1)T_t(k_2 a_1), \\
\text{(c)} & A_2 Q(k_2 a_1, k_2/\sigma_2 c_2^2) & = & B_2 R_t(\sigma_2 a_1, k_2/\sigma_2), \\
\text{(d)} & A_2 \frac{\sigma_2}{k_2}(c_2^2-1)S(k_2 a_1) & = & B_2 \frac{\sigma_2}{k_2}(c_2^2-1)T_t(k_2 a_1).
\end{array}
\right.
$$

The procedure for solving (25) will be the following: Equations (c) and (d) define a one-velocity problem which we solve by the methods of LA-756, taking $A_2=1$ and obtaining $a_1$ and $B_2$ (as functions of $N_1/N_2$) by calculation. Next we calculate $A_1$, as in previous example, from:

$$
(26)\qquad A_1 = \frac{k_1}{k_2}\frac{S(k_2 a_1)}{S(k_1 a_1)}\left[\frac{\sigma_2}{\sigma_1}\frac{N_1}{N_2} - \alpha_1^2\right],
$$

and then $B_1$ from equation (b) above. Finally, equation (a) is used as a test equation for the trial quantity $N_1/N_2$.

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The functionals $R_t(\sigma r, k/\sigma)$ and $T_t(kr)$ correspond to the flux densities (24). Note that the latter have a special form, different from (10), due to the requirement that the neutron flux vanish at infinity. Hence:

$$(27) \quad R_t(\sigma r, k/\sigma) = c\,\frac{\sigma}{k}\,e^{\sigma r}\,E_1\left(\left(1+\frac{k}{\sigma}\right)\sigma r\right); \quad T_t(kr) = -\frac{1+kr}{(kr)^2}\,e^{-kr},$$

with c from $k/\sigma = c\,\mathrm{arth}(k/\sigma)$.

Turning to the first part of the computation, taking 2.4, 2.5, and 2.6 as successive trials for $N_1/N_2$, we calculate:

| | Oralloy | | | Tu |
|---|---|---|---|---|
| | $N_1/N_2$=2.4 | $N_1/N_2$=2.5 | $N_1/N_2$=2.6 | |
| $\bar{c}_{11}, \bar{c}_{12}$ | 1.2440, -.2798 | 1.2440, -.3145 | 1.2440, -.3492 | .98 .62 |
| $\bar{c}_{21}, \bar{c}_{22}$ | 0 1.6958 | 0 1.7305 | 0 1.7652 | 0 .50 |
| $k_1, k_2$ | .34155, .35585 | .34155, .36790 | .34155 .37987 | .08869 .18863 |
| $c_1^1, c_1^2$ | 1.2440, 1.2617 | 1.2440, 1.2763 | 1.2440, 1.2923 | .98, .9035 |
| $c_2^1, c_2^2$ | - 1.6958 | - 1.7305 | - 1.7652 | - .50 |
| $\alpha_1^1, \alpha_1^2$ | 1 -8.532 | 1 -5.255 | 1 -3.902 | 1 -4.374 |
| $\alpha_2^1, \alpha_2^2$ | 0 1 | 0 1 | 0 1 | 0 1 |

These calculations are followed by three one-velocity calculations and the computation of $A_1$ and $B_1$. The results are given in the table below:

[redacted]

| $N_1/N_2$ | $a_1$ | $B_2$ | $A_1$ | $A_1+\alpha_1^2$ | $B_1$ |
|---|---|---|---|---|---|
| 2.4 | 6.129 | .6322 | 9.408 | .876 | 1.345 |
| 2.5 | 5.905 | .5827 | 6.125 | .870 | 1.237 |
| 2.6 | 5.696 | .5403 | 4.785 | .883 | 1.144 |

Finally, testing $N_1/N_2$ by calculating D where D ≡ left-hand side of (25,a) minus the right-hand side, we obtain:

$N_1/N_2=2.4$, $D=-.0518$; $N_1/N_2=2.5$, $D=.0387$; $N_1/N_2=2.6$, $D=.1122$.

Hence, by interpolation $N_1/N_2=2.457$, $a_1=6.001$, $B_2=.6040$, $A_1+\alpha_2^1=.872$, $B_1=1.283$, and by calculation $k_2(0y)=.36274$, $\alpha_1^2(0y)=-6.219$, and $A_1=7.091$.

The average velocity $\bar{v}$ in the core is then given by: $\bar{v} = 3.457 \div \left[(2.457/\sigma_1 v_1) + (1.000/\sigma_2 v_2)\right] = 9.04$ cm/shake to be compared with the Integral Theory result of 8.98, (estimated from the table of $\bar{v}$ vs tamper thickness given in LA-1276).



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## EXAMPLE III

As a final example let us consider the same problem as in Example II but with a finite rather than an infinite tamper. We take the outer radius $a_2$ equal to $2a_1$. The parameters are the same as in Example II but the Tu flux distributions will be different and the boundary conditions more complicated. We have:

$$(28)\qquad \text{Oy:}\quad \begin{cases} \mathcal{N}_1(r) = A_1 \dfrac{\sin k_1 r}{k_1 r} + \alpha_1^2 A_2 \dfrac{\sin k_2 r}{k_2 r} \\[2ex] \mathcal{N}_2(r) = A_2 \dfrac{\sin k_2 r}{k_2 r} \end{cases}$$

$$(29)\qquad \text{Tu:}\quad \begin{cases} \mathcal{N}_1(r) = B_1 \left[ \dfrac{\sinh k_1 r}{k_1 r} + \bar{B}_1 \dfrac{\cosh k_1 r}{k_1 r} \right] + \\[2ex] \qquad\qquad + \alpha_1^2 B_2 \left[ \dfrac{\sinh k_2 r}{k_2 r} + \bar{B}_2 \dfrac{\cosh k_2 r}{k_2 r} \right] \\[2ex] \mathcal{N}_2(r) = B_2 \left[ \dfrac{\sinh k_2 r}{k_2 r} + \bar{B}_2 \dfrac{\cosh k_2 r}{k_2 r} \right] \end{cases}$$

Before writing down the boundary conditions we introduce the following abbreviated notation: We denote $Q(k_i r, k_i/\sigma_g c_g^i)$ by $Q_{ig}(r)$, $\frac{\sigma_g}{k_i}(c_g^i - 1)S(k_i r)$ by $S_{ig}(r)$, and similarly $R(k_i r, k_i/\sigma_g c_g^i)$ by $R_{ig}(r)$ and $\frac{\sigma_g}{k_i}(c_g^i - 1)T(k_i r)$ by $T_{ig}(r)$: The boundary conditions can then be written as:



[redacted]

**Core** **Finite Tamper**

$$
(30)\begin{cases}
(a) & A_1Q_{11}(a_1)+\alpha_1^2A_2Q_{21}(a_1) = B_1Q_{11}(a_1)+\alpha_1^2B_2Q_{21}(a_1)+B_1\bar{B}_1R_{11}(a_1)+\alpha_1^2B_2\bar{B}_2R_{21}(a_1) \\
(b) & A_1S_{11}(a_1)+\alpha_1^2A_2S_{21}(a_1) = B_1S_{11}(a_1)+\alpha_1^2B_2S_{21}(a_1)+B_1\bar{B}_1T_{11}(a_1)+\alpha_1^2B_2\bar{B}_2T_{21}(a_1) \\
(c) & A_2Q_{22}(a_1) = B_2Q_{22}(a_1)+B_2\bar{B}_2R_{22}(a_1) \\
(d) & A_2S_{22}(a_1) = B_2S_{22}(a_1)+B_2\bar{B}_2T_{22}(a_1) \\
(e) & B_1Q_{11}(a_2)+\alpha_1^2B_2Q_{21}(a_2)+B_1\bar{B}_1R_{11}(a_2)+\alpha_1^2B_2\bar{B}_2R_{21}(a_2) = 0 \\
(f) & B_2Q_{22}(a_2)+B_2\bar{B}_2R_{22}(a_2) = 0
\end{cases}
$$

The procedure for solving the above system is very similar to the one outlined in Example II. We take $A_2 = 1$ and solve for $a_1$, $B_2$, and $\bar{B}_2$ by one-velocity methods using (c), (d), and (f). We obtain $A_1$ from (26), then $B_1$ and $\bar{B}_1$ by solving (b) and (e) simultaneously, and finally a check on $N_1/N_2$ by calculating D, where D = left-hand side of (2) minus the right-hand side.

[redacted]

The results of the calculations are summarized in the table below:

| CASE | I | II | III |
|---|---|---|---|
| Calculated Quantities | $N_1/N_2=2.15$ | $N_1/N_2=2.20$ | $N_1/N_2=2.25$ |
| $k_1$, $k_2$ | .34155, .32532 | .34155, .33148 | .34155, .33761 |
| $a_1$ | 6.800 | 6.658 | 6.522 |
| $\alpha_1^2$, $\alpha_1^2 + A_1$ | 5.297, .906 | 9.247, .895 | 25.439, .887 |
| $B_2$, $\bar{B}_2$ | -.7968, -.9990 | -.7599, -.9989 | -.7257, -.9987 |
| $B_1$, $\bar{B}_1$ | -1.8652, -.9033 | -1.7851, -.8998 | -1.7117, -.8954 |
| D | -.0903 | -.0447 | .0035 |

Hence, by interpolation $N_1/N_2=2.246$, $a_1 = 6.532$, $\alpha_1^2+A_1=.888$, $B_2=-.7282$, $\bar{B}_2=-.9987$, $B_1=-1.7170$, $\bar{B}_1=-.8957$, and by calculation $k_2=.33712$, $\alpha_1^2 = 22.541$, and $A_1=-21.653$. For the tamper we have as in Example II $k_1=.08869$, $k_2=.18863$, and $\alpha_1^2=-4.374$.

The average velocity $\bar{v}$ in the core is given by:
$\bar{v} = 3.246 - \left[(2.246/\sigma_1 v_1) + (1.000/\sigma_2 v_2)\right] = 9.24$ compared to the Integral Theory result of 9.14 given in LA-1276. The critical radii also agree well (S.W.: 6.53, I.T.: 6.68), the error being in the direction and of the magnitude one is accustomed to in one-velocity calculations.



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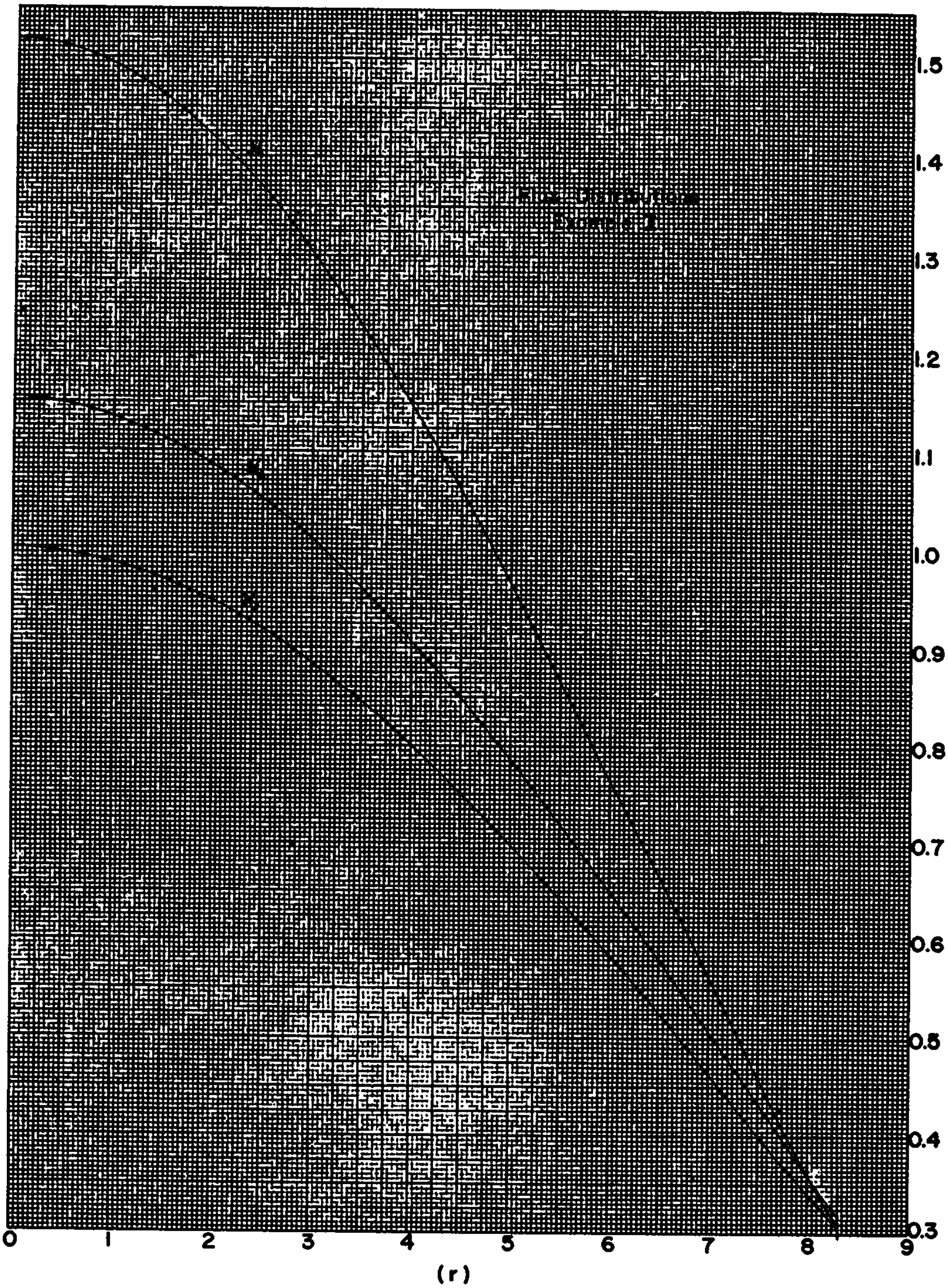




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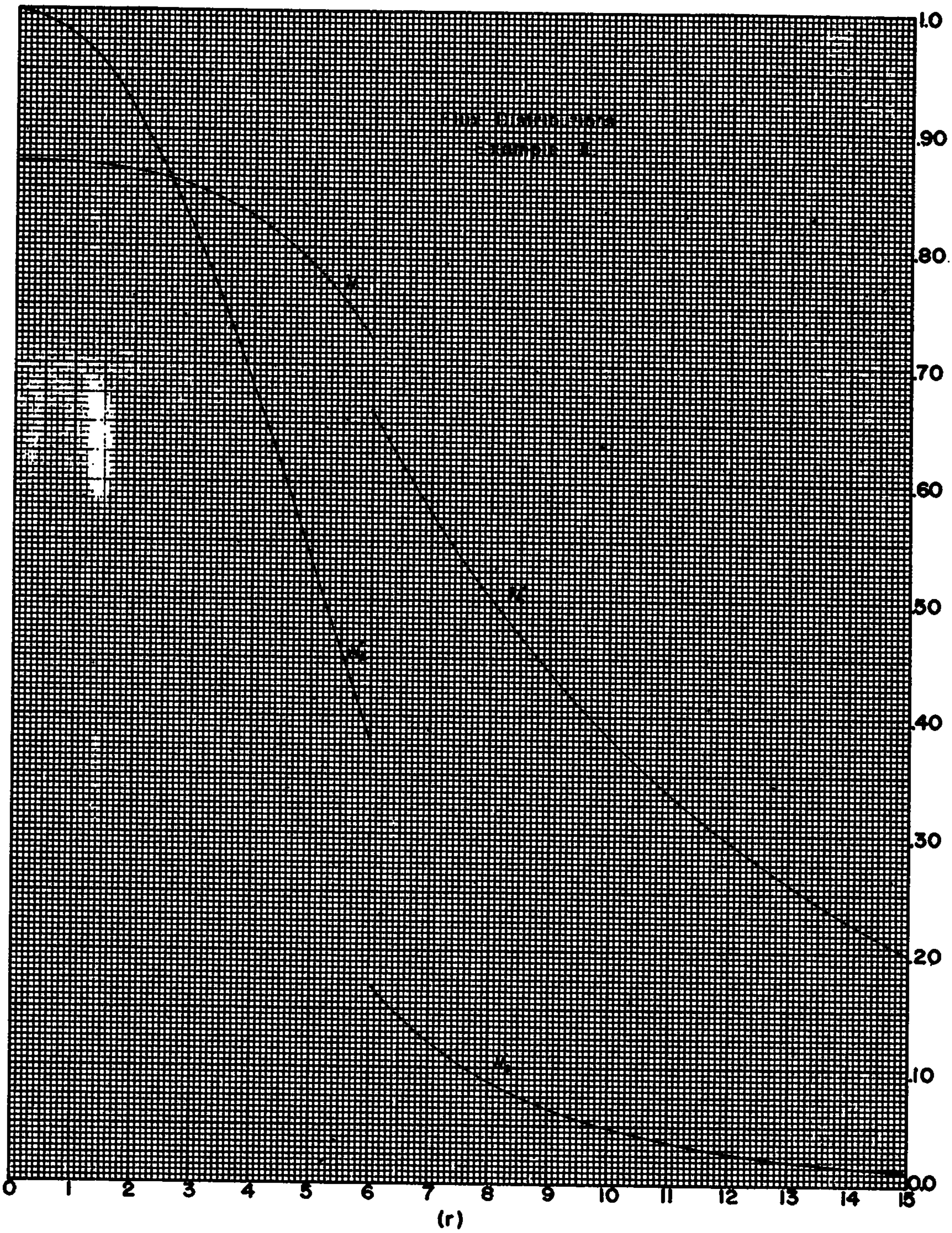




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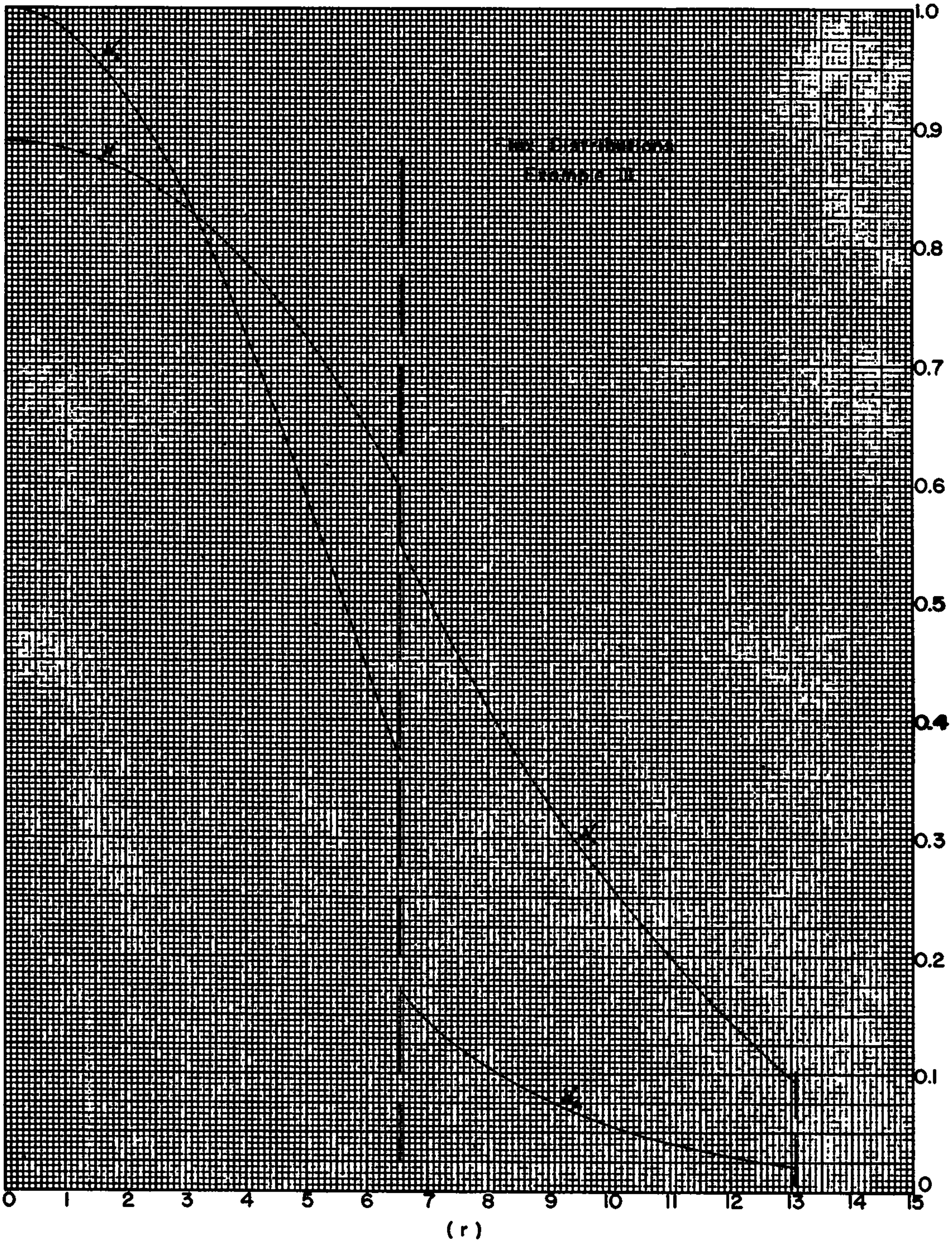




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## APPENDIX

## THE SPHERICAL HARMONIC METHOD

We would like to compare the Serber-Wilson Method with another important method, the Spherical Harmonic Method, both with regard to the complexity of the computations involved, and with regard to the accuracy one may expect. At present, the latter comparison cannot be made since very few multi-velocity calculations have been carried out with either method. In the one-velocity case, however, it is generally held that the Serber-Wilson Method and the $P_3$-Approximation (the Spherical Harmonic Approximation of order three) are, as far as accuracy is concerned, about equivalent.

For the purpose of comparing the two methods with regard to complexity (computational as well as mathematical) the following brief outline of the Spherical Harmonic Method will probably suffice. We consider again the G-velocity isotropic theory and expand the flux distributions $\mathcal{N}_g(r,\mu)$ in Legendre series:

$$(31)\quad \left\{ \begin{aligned} \mathcal{N}_g(r,\mu) &= \frac{1}{2}\sum_{k=0}^{n} (2k+1)\,\psi_{g,k}(r)P_k(\mu), \\ \psi_{g,k}(r) &= \int_{-1}^{1} \mathcal{N}_g(r,\mu)P_k(\mu)d\mu, \quad g = 1, 2, \ldots, G. \end{aligned} \right.$$

where n denotes the degree of approximation and $P_k(\mu)$ the Legendre



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Polynomials. We substitute the above expansion in (9), multiply by $P_\ell(\mu)$, $\ell = 0, 1, \ldots, n$, on both sides and integrate over $\mu$ from -1 to +1. Before performing these integrations, $\mu P_k(\mu)$ and $(1-\mu^2)P_k'(\mu)$ are conveniently replaced by:

$$(32) \qquad \begin{aligned} (2k+1)\mu P_k(\mu) &= \left[ (k+1)P_{k+1}(\mu) + kP_{k-1}(\mu) \right], \\ (2k+1)(1-\mu^2)P_k'(\mu) &= k(k+1) \left[ P_{k-1}(\mu) - P_{k+1}(\mu) \right], \end{aligned}$$

respectively. Carrying out the above steps we arrive at the following system of differential equations:

$$(33) \qquad (k+1)\left(D_r + \frac{k+2}{r}\right) \psi_{g,k+1} + k\left(D_r - \frac{k-1}{r}\right) \psi_{g,k-1} + (2k+1)\sigma_g \psi_{g,k} = \begin{cases} \sum_{h=1}^{G} \sigma_h c_{gh} \psi_{h,0}; & k = 0, \\ 0; & k = 1, 2, \ldots, n, \end{cases}$$

usually referred to as the $P_n$-transform of (9). For reasons which will not be discussed here, n is usually taken to be odd, n = 1, 3, 5, ...

Equations (33) can be written in several alternate forms. For instance, if we let $\psi_{g,k} = \phi_{g,k}/r^{k+1}$, we obtain:

$$(34) \qquad \frac{k+1}{r} D_r \phi_{g,k+1} + k \left[ rD_r - (2k-1) \right] \phi_{g,k-1} + (2k+1) \sigma_g \phi_{g,k} = \begin{cases} \sum_{h=1}^{G} \sigma_h c_{gh} \phi_{h,0}; & k = 0, \\ 0; & k = 1, 2, \ldots, n, \end{cases}$$



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and replacing $r^2$ by x, denoting the right-hand side of (34) by RHS (34), we have:

$$(35)\quad 2(k+1)D_x\,\phi_{g,k+1} + k\left[2xD_x-(2k-1)\right]\phi_{g,k-1} + (2k+1)\sigma_g\,\phi_{g,k} = \text{RHS } (34).$$

To derive the differential equation for $\phi_{g,0}$ for a particular n, the following formula, obtained from (35) by differentiation, is very useful:

$$(36)\quad 2(k+1)D_x^{k+1}\,\phi_{g,k+1} + \frac{1}{2}\,kD_r^2\,D_x^{k-1}\phi_{g,k-1} + (2k+1)\sigma_g D_x^k\,\phi_{g,k} = \text{RHS } (34)$$

For with the aid of (36) we can eliminate the higher order $\phi_{g,k}$'s and be left with a (n+1)-order differential equation in $\phi_{g,0}$ alone.

We obtain, omitting the subscript g for the moment:

$$(37)\quad \begin{cases} 2^2 2!D_x^2\,\phi_2 = -\,D_r^2\,\phi_0 - 3\sigma(2D_x\phi_1), \\ 2^3 3!D_x^3\,\phi_3 = 5\sigma D_r^2\,\phi_0 - (4D_r^2-15\sigma^2)(2D_x\phi_1), \\ 2^4 4!D_x^4\,\phi_4 = (9D_r^2-35\sigma^2)D_r^2\,\phi_0 + \sigma(55D_r^2-105\sigma^2)(2D_x\phi_1), \\ 2^5 5!D_x^5\,\phi_5 = -\sigma(161D_r^2-315\sigma^2)D_r^2\,\phi_0 + (64D_r^4-735\sigma^2D_r^2+945\sigma^4)(2D_x\phi_1), \\ 2^6 6!D_x^6\,\phi_6 = -(25D_r^4-294\sigma^2D_r^2+385\sigma^4)D_r^2\,\phi_0 - \sigma(231D_r^4-1190\sigma^2D_r^2+1155\sigma^4)(2D_x\phi_1), \end{cases}$$

where $2D_x\phi_{g,1} = \left(\sum_{h=1}^{G}\sigma_h c_{gh}\phi_{h,0}\right) - \sigma_g\,\phi_{g,0}$. Since $\phi_{g,n+1} = 0$ in $P_n$-approximation we have the following differential equations for n = 1, 3, and 5:



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$$(38)\quad \begin{cases} P_1: \left[ \dfrac{(3\sigma_g^2 - D_r^2)}{3\sigma_g^2} \right] \phi_{g,0} = \dfrac{1}{\sigma_g} \displaystyle\sum_{h=1}^{G} \sigma_h c_{gh} \phi_{h,0} \\ P_3: \left[ \dfrac{105\sigma_g^4 - 90\sigma_g^2 D_r^2 + 9D_r^4}{\sigma_g^2(105\sigma_g^2 - 55D_r^2)} \right] \phi_{g,0} = \dfrac{1}{\sigma_g} \displaystyle\sum_{h=1}^{G} \sigma_h c_{gh} \phi_{h,0} \\ P_5: \left[ \dfrac{1150\sigma_g^6 - 1575\sigma_g^4 D_r^2 + 525\sigma_g^2 - 25D_r^6}{\sigma_g^2(1155\sigma_g^4 - 1190\sigma_g^2 D_r^2 + 231)D_r^4} \right] \phi_{g,0} = \dfrac{1}{\sigma_g} \displaystyle\sum_{h=1}^{G} \sigma_h c_{gh} \phi_{h,0} \end{cases}$$

where it is understood that the denominator in the brackets operates on the right-hand side and the numerator on $\phi_{g,0}$.

Note that the operators in (38) are continued fraction approximations of z/art z with $z = iD_r/\sigma_g$. For we have:

$$(39)\qquad z/\text{art}\, z = 1 + z^2 \big/ 3+4z^2 \big/ 5+9z^2 \big/ 7+\ldots$$

Denoting the $(n+1)^{th}$ approximation (the first (n+1) terms) of (39) by $\left[z/\text{art}\, z\right]_n$, we have in the general case:

$$(40)\qquad P_n: \left[ (iD_r/\sigma_g) \div \text{art}\,(iD_r/\sigma_g) \right]_n \phi_{g,0} = \frac{1}{\sigma_g} \sum_{h=1}^{G} \sigma_h c_{gh} \phi_{h,0}\ .$$

It can now readily be verified that the general solution of (40) is given by:



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$$(41)\quad \phi_{g,0} = r\,\psi_{g,0} = r\,\mathcal{N}_g(r) = r\sum_{i=1}^{\frac{1}{2}(n+1)G} \alpha_g^i A_i\left[\frac{\sin k_i r}{k_i r} + \bar{A}_i\,\frac{\cos k_i r}{|k_i| r}\right],$$

provided the $k_i$'s and $\alpha_g^i$'s satisfy the following matrix equation:

$$(42)\quad \begin{pmatrix} c_{11} - \left[\frac{k_i/\sigma_1}{\mathrm{art}(k_i/\sigma_1)}\right]_n & c_{12} \cdots\cdots & c_{1G} \\ c_{21} & c_{22} - \left[\frac{k_i/\sigma_2}{\mathrm{art}(k_i/\sigma_2)}\right]_n \cdots & c_{2G} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ c_{G1} & c_{G2} & c_{GG} - \left[\frac{k_i/\sigma_G}{\mathrm{art}(k_i/\sigma_G)}\right]_n \end{pmatrix} \begin{pmatrix} \alpha_1^i \\ \frac{\sigma_2}{\sigma_1}\alpha_2^i \\ \cdot \\ \cdot \\ \cdot \\ \frac{\sigma_G}{\sigma_1}\alpha_G^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

which obviously approaches (11) as n approaches $\infty$. The eigenvalues $k_i$ come in this case from an algebraic equation in $k_i^2$ of degree $\frac{1}{2}(n+1)G$. We can, therefore, expect $\frac{1}{2}(n+1)G$ eigenvalues in the right half of the complex plane.

In comparing the Serber-Wilson and the Spherical Harmonic Methods we observe that the diffiuclty of having a sufficient number of $k_i$'s is replaced by the hardships involved in having to deal with complex ones, and that the Spherical Harmonic Method has $\frac{1}{2}(n+1)$ times as many

$k_i$'s as the Serber-Wilson Method, which implies (for $n \geq 3$) more terms in (41), more $\alpha_g^i$'s to calculate, etc. Finally, the boundary conditions associated with the Spherical Harmonic Method are more difficult to apply, not only because there are more of them ($n \geq 3$), but also because the remaining angular moments, i.e., $\psi_{g,k}$, $k = 1, 2, \ldots, n$, must be computed. For the conditions usually imposed on the $\psi_{g,k}$'s (or $\phi_{g,k}$'s) are that they be continuous at each boundary.

In $P_1$-Approximation we obtain from the first equation in (37):

$$(43) \qquad 2D_x \phi_{g1} = \frac{1}{r} D_r \phi_{g1} = -\frac{1}{3\sigma_g} D_r^2 \phi_{g0} ,$$

and hence:

$$(44) \qquad \psi_{g1} = \sum_{i=1}^{G} \frac{k_i}{3\sigma_g} \alpha_g^i A_i \left[ S_1(k_i r) + \frac{k_i \bar{A}_i}{|k_i|} T_1(k_i r) \right] ,$$

where $S_1(x) = (\sin x - x \cos x)/x^2$ and $T_1(x) = (\cos x + x \sin x)/x^2$.

In $P_3$-Approximation we first solve for $2D_x \phi_{g1}$ in the third equation of (37), then obtaining $2^2 2!\, D_x^2 \phi_{g2}$ and $2^3 3!\, D_x^3 \phi_{g3}$ from the first and second equation:

$$(45) \qquad \left\{ \begin{aligned} 2D_x \phi_{g1} &= \frac{1}{r} D_r \phi_{g1} = -\frac{1}{\sigma_g} \frac{9D_r^2 - 35\sigma_g^2}{55D_r^2 - 105\sigma_g^2} D_r^2 \phi_{g0} , \\ 2^2 2!\, D_x^2 \phi_{g2} &= 2!\, \frac{1}{r} D_r \frac{1}{r} D_r \phi_{g2} = -\frac{28}{55D_r^2 - 105\sigma_g^2} D_r^4 \phi_{g0} , \\ 2^3 3!\, D_x^3 \phi_{g3} &= 3!\, \frac{1}{r} D_r \frac{1}{r} D_r \frac{1}{r} D_r \phi_{g3} = \frac{36/\sigma_g}{55D_r^2 - 105\sigma_g^2} D_r^6 \phi_{g0} . \end{aligned} \right.$$

Solving these differential equations we have:

$$
(46)\quad \begin{cases}
\psi_{g1} = \sum_{i=1}^{2G} \frac{k_i}{\sigma_g} \left[ \frac{35+9k_i^2/\sigma_g^2}{105+55k_i^2/\sigma_g^2} \right] \alpha\, {}_g^iA_i \left[ S_1(k_ir) + \frac{k_i\bar{A}_i}{|k_i|} T_1(k_ir) \right], \\
\psi_{g2} = \sum_{i=1}^{2G} \frac{14k_i^2/\sigma_g^2}{105+55k_i^2/\sigma_g^2} \alpha\, {}_g^iA_i \left[ S_2(k_ir) + \frac{k_i\bar{A}_i}{|k_i|} T_2(k_ir) \right], \\
\psi_{g3} = \sum_{i=1}^{2G} \frac{6k_i^3/\sigma_g^3}{105+55k_i^2/\sigma_g^2} \alpha\, {}_g^iA_i \left[ S_3(k_ir) + \frac{k_i\bar{A}_i}{|k_i|} T_3(k_ir) \right],
\end{cases}
$$

where $S_2(x) = \left[ (3-x^2)\sin x - 3x \cos x \right] / x^3$,

$T_2(x) = \left[ (3-x^2)\cos x + 3x \sin x \right] / x^3$,

$S_3(x) = \left[ (15-6x^2)\sin x - (15-x^2)x \cos x \right] / x^4$, and

$T_3(x) = \left[ (15-6x^2)\cos x + (15-x^2)x \sin x \right] / x^4$.